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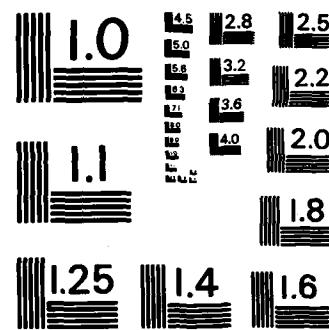
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## Technical Report 796

# SINGULAR PERTURBATIONS OF THE PAIDOUSSIS EQUATION: A THIN CYLINDER IN AN AXIAL FLOW

M. J. Shensa

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Electronics Division

Under authority of

R.H. Hearn, Head  
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)	<p>This report examines, from the viewpoint of singular perturbation theory, a fourth order partial differential equation which was derived by Paidoussis [1] as a model for the behavior of a thin cylinder in an axial flow. It is found that for sufficiently large form drag (<math>c_f &lt; \frac{1}{2}</math>) and small flexural rigidity the influence of the higher order boundary conditions is restricted to the boundary; i.e., the reduced equation is a good approximation. Furthermore for small frequencies the downstream boundary conditions can be ignored in the sense that outside of the very end of the cylinder, their effect on the solution is negligible. Finally, an examination of the characteristics of the reduced PDE leads one to conjecture that this remains true in the case <math>c_f &lt; \frac{1}{2}</math>.</p>	

## SUMMARY

This report examines, from the viewpoint of singular perturbation theory, a fourth order partial differential equation which was derived by Paidoussis [1] as a model for the behavior of a thin cylinder in an axial flow. It is found that for sufficiently large form drag ( $c_t' < \frac{1}{2}$ ) and small flexural rigidity the influence of the higher order boundary conditions is restricted to the boundary; i.e., the reduced equation is a good approximation. Furthermore for small frequencies the downstream boundary conditions can be ignored in the sense that outside of the very end of the cylinder, their effect on the solution is negligible. Finally, an examination of the characteristics of the reduced PDE leads one to conjecture that this remains true in the case  $c_t' \leq \frac{1}{2}$ .

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## I. INTRODUCTION

This report examines, from the viewpoint of singular perturbation theory, a fourth order partial differential equation which was derived by Paidoussis [1] as a model for the behavior of a long thin flexible cylinder in an axial flow. The motivation for our study of this phenomenon as well as a summary of previous work in the area may be found in [2]. In particular, Ortloff and Ives [3] and Kennedy [2,4] succeeded in finding closed form solutions by passing to the limit of zero flexural rigidity, thereby reducing the governing equation to second order. We shall primarily be concerned with the validity of the reduced equation, determining suitable boundary conditions, and assessing the degree of their influence on the solution.

The complete partial differential equation (PDE) is given by (the notation is that of of [2]):

$$\begin{aligned} EIy_{xxxx} + (M+m)y_{tt} + 2MUy_{tx} \\ + (MU^2)(1 - 2c_t(L-x)/d_c - 2c'_t)y_{xx} \\ + (MU^2)(2c_n/\pi d_c + 2c_t/d_c)y_x \\ + (MU)(2c_n/\pi d_c)y_t = 0 \end{aligned} \quad (1.1)$$

where

- $y(x,t)$  = crosstrack position of the cylinder measured at  $x$   
 $x$  = along-track space variable increasing downstream  
 $t$  = time  
 $M$  = virtual mass of fluid per unit,  $\pi\rho(d_c/2)^2$ , where  $\rho$  is the fluid density [1]  
 $m$  = mass of the cylinder per unit length  
 $U$  = free stream velocity  
 $L$  = cylinder length  
 $d_c$  = cylinder diameter  
 $c_t$  = tangential (longitudinal) drag coefficient  
 $c'_t$  = drag coefficient added to account for a drogue and/or form drag at the end of the cable  
 $c_n$  = coefficient of the linearized normal drag expression  
 $E$  = Young's modulus  
 $I$  = moment of inertia =  $\pi/4(d_c/2)^4$ .

- 
1. M. P. Paidoussis, "Dynamics of Flexible Slender Cylinders in Axial Flow," Journal of Fluid Mechanics, Vol 26, 1966, pp 717-751.
  2. R. Kennedy and E. Strahan, "A Linear Theory of Transverse Cable Dynamics at Low Frequencies," Naval Underwater Systems Center TR 6463, June 1981.
  3. C. R. Ortloff and J. Ives, "On the Dynamic Motion of a Thin Flexible Cylinder in a Viscous Stream," Journal of Fluid Mechanics, Vol 38, part 4, 1969, pp 713-720.
  4. R. M. Kennedy, "Low Frequency Transverse Motion Response of Cable Towed Array Systems," NUSC Technical Memorandum No. 93231, Naval Underwater Systems Center, New London, CT, July 1979.

The cylinder is driven by a forcing function  $y(0,t)$  at the "tow point" as pictured in Figure 1.

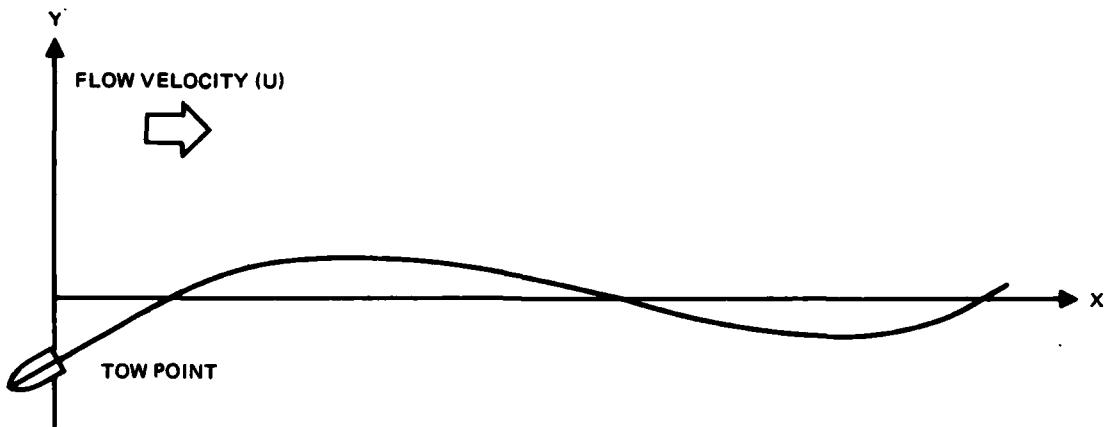


Figure 1. Physical model approximated by equation (1.1).

Several approximations were involved in the derivation [1] of equation (1.1). These include a hydrodynamic approximation which uses the virtual mass  $M = \pi\rho(d_c/2)^2$  concept (valid for a straight cylinder) to calculate the normal component of the fluid momentum of a curved cylinder, and a mathematical approximation which assumes  $y_x$ , the tangent of the angle of the cylinder with the x-axis, to be sufficiently small. The latter is reflected in several places: the neglect of nonlinear drag terms in the first order derivatives, the term "1" in the coefficient of  $y_{xx}$  which approximates the cosine of the angle of cylinder's axis, and the use of  $x = L$  for the end point of the cylinder.

A dimensionless version of (1.1) is given by

$$\epsilon^3 h \eta_{\xi\xi\xi\xi} + \epsilon \eta_{\tau\tau} + 2\beta \epsilon \eta_{\tau\xi} + b(\xi - 1 - \epsilon q)\eta_{\xi\xi} + (b + d)\eta_{\xi} + d\eta_{\tau} = 0 \quad (1.2a)$$

where

$$\xi = \frac{x}{L}, \tau = \frac{U}{L} t, \eta(\xi, \tau) = \frac{y(x, t)}{\Delta} \quad (1.2b)$$

$$\Delta = \text{forcing function amplitude} \quad (1.2c)$$

$$\epsilon = d_c/L \quad (1.2d)$$

$$\beta = M/(m + M) \quad (1.2e)$$

$$h = \frac{1}{16} \frac{E\beta}{\rho U^2} \quad (1.2f)$$

$$b = 2\beta c_t \quad (1.2g)$$

$$q = (2c'_t - 1)/2c_t \quad (1.2h)$$

$$d = 2\beta c_n/\pi \quad (1.2i)$$

In the study that follows, we shall be interested in the behavior of the solution as  $\epsilon^3 h \rightarrow 0$ . One way to achieve this is to let the relative cylinder diameter approach zero; however, it should be noted that  $\beta$  depends on  $d_c$  through  $M$  and, possibly, through  $m$ . We shall assume

that the mass of the cylinder per unit length is proportional to  $d_c^2$  so that  $\beta = \text{constant}$ . This includes the primary case of interest, that of neutral buoyancy for which  $m = \rho\pi(d_c/2)^2$ , and thus  $\beta = \frac{1}{2}$ .

The boundary conditions of (1.2) are those of a cylinder pinned at one end ( $\xi = 0$ ) and free at the other end ( $\xi = 1$ ). The boundary conditions at the pinned end are provided by a forcing function  $\phi_1$

$$\eta(0, \tau) = \phi_1(\tau) \quad (1.3)$$

and a requirement that the bending moment be zero [1]

$$\eta_{\xi\xi}(0, \tau) = 0 \quad . \quad (1.4)$$

Similarly, there must be no bending moment at the free end

$$\eta_{\xi\xi}(1, \tau) = 0 \quad . \quad (1.5)$$

The second boundary condition represents a balancing of the lateral shear force and is given by the somewhat complex expression [1]

$$EI y_{xxx} + fMU(y_t + Uy_x) - (m + fM)x_e y_{tt} = 0 \quad (1.6)$$

where  $f$  is a constant less than 1 and

$$x_e = \frac{1}{A} \int_{L-\ell}^L A(x) dx \quad (1.7)$$

with  $A$  the cross-sectioned area for an end tapered from  $L - \ell$  to  $L$ .

To simplify our computations we shall take  $f = 1$  and assume that  $\ell \ll L$  so that  $x_e = 0$ . However, note that in the case of a drogue, which in [2] is modeled as an extra piece of cylinder,  $x_e$  is not zero. Alternatively, a drogue may be considered a point mass  $m_D$  so that the last term is replaced by the acceleration  $m_D y_{tt}$ . In dimensionless units this gives

$$h\epsilon^2 \eta_{\xi\xi\xi} + \beta(\eta_\tau + \eta_\xi) + \alpha\beta\eta_{tt} = 0 \quad (1.8)$$

where  $\alpha = \text{mass of drogue}/\text{mass of cylinder at neutral buoyancy}$ . (More precisely,  $\alpha = m_D/ML$ .)

As the flexural rigidity goes to zero, (1.8) becomes an expression of the first order in  $\xi$ . As will be detailed in section II, this implies that the boundary condition

$$(\eta_\tau + \eta_\xi) + \alpha\eta_{tt} = 0 \quad , \quad (1.9)$$

derived from (1.8), is appropriate for the reduced equation. Thus, one must resist the tendency to dismiss or remain indifferent to (1.6) simply because it is the higher order boundary condition.

Since the problem as stated here is linear, we shall find it expedient in a great portion of the development to use separation of variables. For this purpose  $\eta$  is allowed to assume complex values, and it is understood that the physical solution is given by its real part. The driving force is taken as  $\phi_1(\tau) = e^{i\tau\omega}$  so that real  $(\phi_1) = \cos(\omega\tau)$ , and more general solutions

may be obtained by super-position. We use this approach in section II and solve the resulting ordinary differential equation (ODE) by singular perturbation analysis. Many of the specific computations rest on the assumption  $\omega \ll 1$ ; i.e., are for large wavelengths. However, this does not preclude (and in fact would even indicate) the qualitative validity of our results in a more general context. The mathematically crucial assumption that  $c'_t > \frac{1}{2}$  is abandoned in section III, but we are still able to derive several interesting although incomplete results. In particular, the reduced PDE (as opposed to ODE of section II) is examined by studying its characteristics. Only the "steady state" solution is considered; aspects relating to initial conditions and/or stability are conveniently ignored. Our methodology shall not be uniformly rigorous but is intended to give a reasonably logical analysis while pointing out the various gaps that remain to be filled.

## II. A SINGULAR PERTURBATION ANALYSIS FOR THE CASE $c'_t > \frac{1}{2}$

This section is devoted to a singular perturbation analysis in which the effects of the structural rigidity, as reflected in the coefficient  $\epsilon^3 h$  of eq. (1.2a), are small. Following O'Malley [5], we define a perturbation parameter

$$\mu = \sqrt{\epsilon^3 h} \quad (2.1)$$

and try to find an asymptotic expansion in that parameter. Note that equation (1.2a) really involves two small parameters,  $\mu$  and  $\epsilon$ , corresponding to the structural rigidity and relative cylinder diameter, respectively. This aspect shall be addressed in more detail later.

We separate variables making the substitution

$$\eta(\xi, \tau) = e^{i\omega\tau} v(\xi) \quad (2.2)$$

and obtain the fourth order ordinary differential equation (cf., (A.1), (A.4), and (A.5))

$$\mu^2 v^{(4)} + \alpha_1(\xi - 1 - \epsilon q) v^{(2)} + \alpha_2 v^{(1)} + \alpha_3 v = 0 \quad (2.3a)$$

where  $0 \leq \xi \leq 1$  and

$$\begin{aligned} \alpha_1 &= b \\ \alpha_2 &= b + d + 2\beta\epsilon i\omega \triangleq b(1 - \nu) \\ \alpha_3 &= \frac{4\omega}{d} \left( \frac{id}{b} - \frac{\epsilon\omega}{b} \right) \triangleq b_0^2 \end{aligned} \quad (2.3b)$$

The notation  $v^{(j)}$  is used to represent the  $j^{\text{th}}$  derivative of  $v$  with respect to  $\xi$ .

Following [5], we seek a composite asymptotic expansion of the form

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5. R. O'Malley, Jr., Introduction to Singular Perturbations, Academic Press, NY, 1974, Chapters 3 and 8.

$$\begin{aligned}
v(\xi, \mu) = & G_1(\xi) + G_2(\xi) + \mu[G_{11}(\xi) + G_{21}(\xi)] + \dots \\
& + \mu^2[L_0(\xi) + \mu L_1(\xi) + \dots] e^{-g_1(\xi)/\mu} \\
& + \mu^2[R_0(\xi) + \mu R_1(\xi) + \dots] e^{-g_2(\xi)/\mu}
\end{aligned} \tag{2.4}$$

with boundary conditions

$$v(0, \mu) = 1 \tag{2.5a}$$

$$v_{\xi\xi}(0, \mu) = 0 \tag{2.5b}$$

$$\mu^2 v^{(3)}(1, \mu) + \beta e(v_{\xi}(1, \mu) + i\omega v(1, \mu)) = 0 \tag{2.5c}$$

$$v_{\xi\xi}(1, \mu) = 0 \tag{2.5d}$$

The first line of equation (2.4) contains the so-called reduced solution  $G_1(\xi) + G_2(\xi)$  of (2.2), obtained by setting  $\mu = 0$ . Since the order of the resulting equation is 2, there are two linearly independent solutions,  $G_1$  and  $G_2$ , determined by the boundary conditions (2.5a) and (2.5c). The functions  $G_{ij}$  are the higher order terms in the expansion modulo the "boundary layer" terms found in the next two lines. For small  $\mu$  the exponential terms  $e^{-g_1/\mu}$  and  $e^{-g_2/\mu}$  are significant only in a small area, so-called inner regions, about the left hand and right hand boundaries, respectively. The factors of  $\mu^2$  preceding these terms are dictated by the boundary conditions (2.5b) and (2.5d) in which the second derivatives of  $v$  produce a factor  $(1/\mu)^2$  which must be canceled by the coefficient  $\mu^2$ .

The boundary conditions (2.5a) – (2.5d) were obtained from (1.3), (1.4), (1.5), and (1.6) – (1.8) through the separation of variables (2.2). Note that the form of (2.2), which in conjunction with (2.5a) also implies the boundary condition  $\eta(0, \tau) = e^{i\omega\tau}$ , represents no real restriction since equation (1.2a) is linear in  $\eta$  and we may apply superposition to construct a more general solution. For computational simplicity, we have set  $\alpha = 0$  in equation (1.8). Actually, this should be a valid approximation (cf. section I) in the absence of a drogue, with a drogue of sufficiently small mass, or for very small  $\omega$ .

The unknown functions of (2.4) are determined by substituting (2.4) into (2.3) and equating the coefficients of powers of  $\mu$  (or of  $\mu j e^{-g/\mu}$  since  $e^{-g/\mu}$  approaches zero faster than any power of  $\mu$ ) to zero. This leads to a sequence of differential equations whose constants are determined by the boundary conditions (2.5). At stage  $j$  a set of simultaneous equations (2.5a) and (2.5c) specify  $G_{1j}$  and  $G_{2j}$ .  $L_j$  and  $R_j$  are then determined from (2.5b) and (2.5d) respectively. It should be noted that the coefficient of  $\mu^j$  in (2.5c) only involves  $R_k$  for  $k < j$  since the coefficient of  $R_j$  involves powers of  $\mu$  of the order  $2 + (2 + j - 3) = j + 1$  and higher. We begin our computations with the determination of  $G_1$  and  $G_2$ , the reduced problem.

## REDUCED PROBLEM

The reduced problem, obtained by setting  $\mu = 0$  in (2.3) is given by

$$\alpha_1(\xi - 1 - \epsilon q) \tilde{v}^{(2)} + \alpha_2 \tilde{v}^{(1)} + \alpha_3 \tilde{v} = 0 \tag{2.6a}$$

with

$$\tilde{v}(0) = 1 \tag{2.6b}$$

$$\tilde{v}_g(1) + i\omega \tilde{v}(1) = 0. \quad (2.6c)$$

This statement is somewhat unusual in that the boundary condition (2.6c) is derived from the highest order condition of the full equation. (It occurs because (2.5c) involves  $\mu$ , and its reduced order is less than that of (2.5d).) Nevertheless, such behavior does not appear to create any problems, and suitable generalizations along the lines of [6]–[7] should lead to the standard theorem [5]

$$\lim_{\mu \rightarrow 0} v(\mu, \xi) = \tilde{v}(\xi) \text{ in } (0,1) \quad (2.7)$$

*provided* the coefficient of  $\tilde{v}^{(2)}$  in (2.6a) is uniformly bounded away from zero; i.e., provided

$$\epsilon q \geq \delta > 0. \quad (2.8)$$

It might seem that  $q > 0$  would constitute a sufficient condition for (2.8) to hold, but some extra care is warranted. Recall that  $\mu = (he^3)^{1/2}$ . Strictly speaking, the limit in (2.7), as described in [5], assumes that  $\epsilon$  is a constant since it also appears in several coefficients of (2.3): we have included  $\epsilon$  in (2.8) to emphasize this point. One would strongly suspect, however, that (2.7) remains valid as long as (2.8) is true even if  $\epsilon \rightarrow 0$  (and thus  $q \rightarrow \infty$ ) although a rigorous demonstration would require a careful examination of the proofs appearing in [5] – [7].

A closed form solution to (2.6a) was derived in [3] in terms of Bessel functions [8], which we now write in the following form (cf. Appendix, eqs. (A.9)–(A.10)):

$$G_1 + G_2 = AF_1(b_0^2 z^2) + BF_2(b_0^2 z^2) \quad (2.9)$$

where A and B are constants,

$$F_1(b_0^2 z^2) = \sum_{m=0}^{\infty} \frac{(-1)^m \left( \frac{b_0^2 z^2}{4} \right)^m}{m! \Gamma(-\nu + m + 1)}, \quad (2.10)$$

$$F_2(b_0^2 z^2) = \left( \frac{b_0^2 z^2}{4} \right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m \left( \frac{b_0^2 z^2}{4} \right)^m}{m! \Gamma(\nu + m + 1)}, \quad (2.11)$$

$$z^2 = b(\xi - 1 - \epsilon q) . \quad (2.12)$$

and  $\nu$  and  $b_0^2$  are given by (A.4) and (A.5). The reduced boundary conditions (2.6b) and (2.6c) become

$$AF_1(b_0^2 z^2) + BF_2(b_0^2 z^2) = 1 \quad (2.13)$$

$$Ab_0^2 b F'_1(b_0^2 z_1^2) + Bb_0^2 b F'_2(b_0^2 z_1^2) + i\omega [AF_1(b_0^2 z_1^2) + BF_2(b_0^2 z_1^2)] = 0 \quad (2.14)$$

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- 6. G. Birkhoff, "On the Asymptotic Character of the Solutions of Certain Linear Differential Equations," *Trans. AMS.*, Vol 9, April 1908, pp 219–230.
  - 7. Turritin, H., "Asymptotic Solutions of Certain Ordinary Differential Equations Associated with Multiple Roots of the Characteristic Equation," *Amer. J. Math.*, Vol 58, 1936, pp 364–378.
  - 8. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press, 1966, p 97.

with

$$z_0^2 = -b(1 + \epsilon q) \quad (2.15a)$$

$$z_1^2 = -b\epsilon q \quad (2.15b)$$

and  $F'$  indicates the derivative of the function  $F$  with respect to its argument.

Since  $\Gamma(r+1) = r\Gamma(r)$ , the ratios of the second terms to the first terms in the series expansions of  $F_i$  and  $F'_i$  (eqs. (A.9)–(A.12)) are bounded by (real part of  $\nu$  negative)

$$\left| \frac{b_0^2 z^2}{4} \right| \frac{1}{|1-\nu|} \quad (2.16)$$

Thus, for sufficiently small  $|bb_0^2\epsilon q|$  (and  $\nu$  not too close to an integer), we may approximate (2.14) by substituting the first terms of the power series of  $F_i$  and  $F'_i$

$$\begin{aligned} A & \left[ \frac{-b_0^2 b}{4\Gamma(-\nu+2)} + \frac{i\omega}{\Gamma(-\nu+1)} \right] \\ & + B \left[ \frac{-b_0^2 b\nu}{4\Gamma(\nu+1)} \left( -\frac{1}{4} bb_0^2 \epsilon q \right)^{\nu-1} + \frac{i\omega}{\Gamma(\nu+1)} \left( -\frac{1}{4} bb_0^2 \epsilon q \right)^\nu \right] = 0 \quad . \end{aligned} \quad (2.17)$$

Some typical values of the parameters involved are [2]  $c_t = .01$ ,  $c_n = .2$ ,  $\epsilon = 10^5$ , and  $c'_t < 200$ . In that case we see that for  $\omega \ll 10^4$  (from eqs. (A.4) and (A.5))

$$bb_0^2 \sim -4i\omega\nu \quad . \quad (2.18)$$

Also, expression (2.16), whose value is approximately  $|\omega\epsilon q|$  for  $z = z_1$  and  $|\omega|$  for  $z = z_0$ , is small for

$$\omega \ll 1 \quad . \quad (2.19)$$

Substituting (2.18) in (2.17) yields

$$A \left[ \frac{-\nu}{\Gamma(-\nu+2)} + \frac{1}{\Gamma(-\nu+1)} \right] + B \left[ \frac{1}{\Gamma(\nu+1)} (i\omega\nu\epsilon q)^{\nu-1} (\nu^2 + i\omega\nu\epsilon q) \right] = 0 \quad . \quad (2.20)$$

and after some manipulation the approximation  $\nu^2 + i\omega\nu\epsilon q \sim \nu^2$  gives

$$\frac{B}{A} \sim \frac{-2\Gamma(\nu+1)}{\nu^2(-\nu+1)\Gamma(-\nu+1)} (i\omega\nu\epsilon q)^{1-\nu} \quad . \quad (2.21)$$

For the above values, as long as  $\nu$  is not close to a negative integer,  $|B| \ll |A|$ . However, the physically significant quantity is not the relative size of the coefficients, but rather that of the functions  $G_1 = AF_1$  and  $G_2 = BF_2$ . Taking the ratios of these two functions, we find from (2.10), (2.11), and (2.21) that

$$\begin{aligned} \left| \frac{G_2(z)}{G_1(z)} \right| &\sim \left| \frac{B}{A} \left( \frac{b_0^2 z^2}{4} \right)^\nu \frac{\Gamma(-\nu + 1)}{\Gamma(\nu + 1)} \right| \sim \left| \frac{2}{\nu^2(1-\nu)} \right| \left| \omega \nu \epsilon q \right|^{1-\nu} \left| \frac{b_0^2 z^2}{4} \right|^\nu \\ &\leq \left| \frac{2}{\nu^2(1-\nu)} \right| \left| \omega \nu \epsilon q \right| \end{aligned} \quad (2.22)$$

Since real ( $\nu$ )  $< 0$  and  $|z|^2$  is a minimum for  $z^2 = z_1^2$ . Thus we find that for  $\omega \ll 1$

$$|G_2(\xi)| \ll |G_1(\xi)| \quad \xi \text{ in } [0, 1] . \quad (2.23)$$

In fact, for the above values,  $\nu \sim -7$  and  $\epsilon q \sim .2$  so that

$$|G_2(\xi)| < .01 \omega |G_1(\xi)| . \quad (2.24)$$

Intuitively, what has occurred is that  $F'_2$  is more singular than  $F_2$  at  $z = z_1$ ; however,  $F'_1$  and  $F_1$  are about the same magnitude. The boundary condition (2.14) forces  $BF'_2$  to be of the same order as  $AF'_1$ ; and thus,  $G_2 = BF_2$  is much smaller than  $G_1 = AF_1$ . When it is appropriate to neglect  $G_2$  (e.g., when  $\omega \ll 1$ ), we may write the two term approximation of (2.13) as

$$A \left[ \frac{1}{\Gamma(-\nu + 1)} - \left( \frac{b_0^2 z_0^2}{4} \right) \frac{1}{\Gamma(-\nu + 1)} \right] \sim 1 \quad (2.25)$$

and the one term (zeroth order) as

$$A \sim \Gamma(-\nu + 1) . \quad (2.26)$$

## THE INNER REGION EXPANSIONS

In this section we seek expressions for the functions  $g_1$ ,  $g_2$ ,  $L_i$ , and  $R_i$  of equation (2.4). They are determined by substituting (2.4) into equation (2.3) and matching powers of  $\mu$ . A straightforward computation (Appendix A) reveals that the most singular term, involving  $\mu^{-2} e^{-g/\mu}$ , vanishes provided

$$g = \pm \frac{2}{3} \sqrt{\alpha_1} (-x)^{3/2} + \text{constant} \quad (2.27)$$

where  $x = \xi - 1 - \epsilon q$ . In order that our expressions remain bounded in  $[0, 1]$  and still produce a finite contribution at the boundaries as  $\mu \rightarrow 0$  we must have  $g_1(\xi = 0) = g_2(\xi = 1) = 0$ , and hence

$$g_1(\xi) \equiv g_1(x(\xi)) = \frac{2}{3} \sqrt{\alpha_1} \left[ (1 + \epsilon q)^{3/2} - (-x)^{3/2} \right] \quad (2.28a)$$

$$g_2(\xi) \equiv g_2(x(\xi)) = \frac{2}{3} \sqrt{\alpha_1} \left[ (-x)^{3/2} - (\epsilon q)^{3/2} \right] . \quad (2.28b)$$

Note that we have taken the liberty of using the same symbol,  $g$ , to represent two different functions,  $g(\xi)$  and  $g(x)$ . The dependency  $g(x)$  is assumed unless otherwise indicated by the context. Also  $x$  is not the same as the variable "x" which appears in equations (1.1) and (1.2b).

Substituting these expressions into (2.3) and equating the coefficient of  $\mu^{-2}e^{-g/\mu}$  to zero yields (eq. (A-24))

$$R'_0 + \frac{1}{2x} \left( \frac{5}{2} - \frac{\alpha_2}{\alpha_1} \right) R_0 = 0 \quad (2.29)$$

which has the solution

$$\begin{aligned} R_0 &= r_0(-x)^2 \left( \frac{\alpha_2}{\alpha_1} - \frac{5}{2} \right) \\ &= r_0(-x)^2 \left( -\nu - \frac{3}{2} \right) \end{aligned} \quad (2.30)$$

with exactly the same equations holding for  $L_0$ . The constant  $r_0$  is determined by satisfying the boundary condition (2.5d) up to order  $O(1)$  in  $\mu$  for the expansion (2.4). Making use of equation (A.16) for the second derivative of  $R_0 e^{-g_2/\mu}$ , we have up to  $O(1)$  in  $\mu$

$$G''_1(1) + G''_2(1) + r_0(g'_2(-\epsilon q))^2 R_0(-\epsilon q) = 0 \quad (2.31)$$

(Recall  $G$  is a function of  $\xi$ , whereas  $g_2$  is a function of  $x$ .)

Once again making the assumption that  $|bb'_0\epsilon q|$  is sufficiently small, we may obtain an approximate solution of (2.31) for  $r_0$ . From (2.18), (2.21), (A.11b), and (A.12b) we find that

$$\begin{aligned} \frac{G''_2(1)}{G''_1(1)} &\sim \frac{-2\Gamma(\nu+1)}{\nu^2(-\nu+1)\Gamma(-\nu+1)} (i\omega\nu\epsilon q)^{1-\nu} \frac{F''_2}{F''_1} \\ &= \frac{2}{\nu} \frac{\Gamma(-\nu+3)}{\Gamma(-\nu+1)} (i\omega\nu\epsilon q)^{-1} \\ &\sim 2\nu(i\omega\nu\epsilon q)^{-1} \\ &= \frac{2}{i\omega\epsilon q} \end{aligned} \quad (2.32)$$

and, recalling that  $G''_1 = (b'_0 b)^2 F''_1$ ,

$$G''_2(1) \sim \frac{2A\nu^2}{\Gamma(-\nu+3)} \frac{i\omega}{\epsilon q} \quad (2.33)$$

Thus, we may ignore  $G''_1(1)$  in (2.31) and approximate  $G''_2(1)$  by (2.33). Also, since  $\alpha_1 = b$

$$g'_2(-\epsilon q) = -\sqrt{b} \sqrt{\epsilon q} \quad (2.34)$$

and

$$R_0(-\epsilon q) = r_0(\epsilon q)^2 \left( -\nu - \frac{3}{2} \right) . \quad (2.35)$$

Hence,

$$\frac{2A\nu^2}{\Gamma(-\nu+3)} \frac{i\omega}{\epsilon q} + r_0(b\epsilon q)(\epsilon q)^{-\frac{1}{2}} \left( \nu + \frac{3}{2} \right) \sim 0 \quad (2.36)$$

or

$$r_0 \sim - \frac{2A\nu^2}{\Gamma(-\nu+3)} \frac{i\omega}{b} (\epsilon q)^{\frac{1}{2}} - \frac{5}{4} . \quad (2.37)$$

As in the case of the reduced equation, we are interested in the contribution of  $\mu^2 R_0 e^{-g_2/\mu}$  relative to the entire solution. Hence, we examine the ratio

$$\begin{aligned} \left| \frac{\mu^2 R_0 e^{-g_2/\mu}}{G_1} \right| &\sim \left| \frac{2\nu^2}{\Gamma(-\nu+3)} \frac{\Gamma(-\nu+1)\omega}{b} (\epsilon q)^{\frac{1}{2}} - \frac{5}{4} \right| \cdot \mu^2 \\ &\quad \cdot \left| (-x)^{-\frac{\nu}{2}} - \frac{3}{4} e^{-g_2/\mu} \right| \\ &\sim \left| \frac{2\mu^2}{b} \omega(\epsilon q)^{\frac{1}{2}} - \frac{5}{4} \right| \left| (-x)^{-\frac{\nu}{2}} - \frac{3}{4} e^{-g_2/\mu} \right| . \end{aligned} \quad (2.38)$$

Inasmuch as the perturbation parameter  $\mu$  is assumed small, the ratio (2.38) will be significant outside the right hand "boundary region" (i.e., a neighborhood of  $x = -\epsilon q$ ) only if  $\epsilon q$  is very small. Furthermore, since  $\mu = \sqrt{\epsilon^3 h}$  the exponential in (2.38) dominates as  $\epsilon$  goes to zero, and

$$\lim_{\epsilon \rightarrow 0} [\text{ratio (2.38)}] = 0 \quad \text{for } x \leq -\epsilon q . \quad (2.39)$$

Thus,  $q$  itself will have to be very small if (2.38) is to be  $O(1)$  outside the boundary region. To be precise, let us, for example, set the edge of the boundary region to be at  $x = -\epsilon q - 0.1$  and determine the magnitude of  $q$  necessary for (2.38) to die out outside the region. As earlier, we take nominal values of  $\nu \sim -7$ ,  $\epsilon = 10^{-5}$ , and  $b = .01$ . For small  $\epsilon q$ ,  $g_2(-\epsilon q - .1) \sim \sqrt{b}(.1)^{3/2} \sim 2 \cdot 10^{-3}$ . Then (2.38) will be  $O(1)$  if  $\epsilon q \ll 1$  and

$$200\omega(\epsilon q)^{-\frac{1}{4}} (.1)^{\frac{11}{4}} \mu^2 e^{-0.002/\mu} \sim 1 . \quad (2.40)$$

If Young's Modulus  $E$  is about  $10^{10}$ ,  $U = 500$ , and  $\rho = 1$  in cgs units [2], then  $h = \beta E / (16\rho U^2) \sim 1/8 \cdot 10^4$  so that  $\mu \sim 3 \cdot 10^{-13/2}$ . Substituting these values into (2.40), we obtain (since  $\omega \ll 1$ )

$$q < \frac{1}{10^{500}} ; \quad (2.41)$$

i.e.,  $q$  has to be essentially zero for the "inner" solution<sup>†</sup> to have a significant effect (except at the very boundary).

## REMARKS ON CONVERGENCE

For  $\epsilon q = 0$ , the coefficient  $r_0$  of (2.37) becomes infinite. It is thus natural to inquire as to conditions (i.e., how small an  $h$  is necessary for a given  $\epsilon$  and  $q$ ) such that the first term of the asymptotic expansion is a good approximation to the true solution. Such a study should be straightforward, although we shall not undertake it in the present paper.

An interesting, but more difficult question, is the validity of (2.4) as an asymptotic expansion in  $\epsilon$ . Some preliminary computations lead us to conjecture that the expansion is not uniform in  $\epsilon$  as  $\epsilon \rightarrow 0$  (with  $h$  fixed) on the full interval  $\xi \in [0, 1]$ , but it is uniform on any closed proper subinterval  $[0, 1 - \delta]$  where  $0 < \delta$ . (There should be no problem at the left hand boundary.) One might also consider various combinations of  $h$ ,  $\epsilon$ , and  $q$  simultaneously approaching zero and determine conditions on their relative magnitudes which assure the asymptotic convergence of expansion (2.4).

## CONCLUSIONS

For  $q > 0$  (i.e.,  $c'_t > 1/2$ ) and small flexural rigidity ( $h \rightarrow 0$ ) the influence of the higher order boundary conditions is restricted to the boundary. In other words, the reduced equation is a good approximation. This seems to imply that the appropriate downstream boundary condition in the absence of flexural rigidity is  $\eta_r + \eta_\xi|_{\xi=1} = 0$  or, more generally, (1.9). This result does not appear obvious from direct physical considerations. Furthermore, for small frequencies (condition 2.19), we can ignore all right hand boundary conditions in the sense that, outside of the very end of the cylinder, the effect of the reduced boundary condition (1.9) on the solution is negligible (equation (2.23)). The reader is warned, however, that these comments are all qualitative. For a more precise statement, he/she should examine the mathematical details which have been presented throughout this section.

## III. THE CASE $c'_t \leq 1/2$ (A POTPOURRI OF CONJECTURES)

The analysis of the previous section breaks down when  $c'_t \leq 1/2$  (i.e.,  $q \leq 0$ ). In that case the ODE (2.3) obtained by separation of variables possesses a turning point at  $\xi = 1 + \epsilon q$  so that the composite expansion (2.4) is no longer valid, and the possibility of extremely anomalous behavior cannot be ruled out ([5], Chapter 8). Furthermore, an examination on the reduced PDE (obtained by setting  $h = 0$  in (1.2a)) reveals that it changes from hyperbolic type to elliptic at  $\xi = \beta^2 \epsilon / b + 1 + \epsilon q$ ; hence, an elliptic region will exist if  $q < -\beta^2 / b$ . Of major concern, then, is a consistent formulation of the problem: What are the appropriate boundary conditions for the PDE (1.2a)? Is the second order reduced equation a valid approximation to (1.2a)? and What are suitable boundary conditions for the reduced PDE? A detailed investigation of these issues is beyond the scope of the present study. Thus we shall try to briefly analyze the situation, point out some of the more salient features, and close with a list of conjectures and open questions.

Let us begin with an examination of the reduced PDE

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<sup>†</sup>We use the term "inner solution" to denote  $\mu^2(R_0 + \mu R_1 + \dots)e^{-g_2/\mu}$  although strictly speaking we are referring to the part of the composite expansion associated with the inner solution.

$$\epsilon\eta_{\tau\tau} + 2\beta\epsilon\eta_{\tau\xi} + b(\xi - 1 - \epsilon q)\eta_{\xi\xi} + (b + d)\eta_\xi + dr_\tau = 0 \quad (3.1)$$

which, for convenience, we rewrite as

$$\eta_{\tau\tau} + 2\beta\eta_{\tau\xi} + b\gamma\eta_{\xi\xi} + \frac{(b+d)}{\epsilon}\eta_\xi + \frac{d}{\epsilon}\eta_\tau = 0 \quad (3.2a)$$

where the coefficient  $\gamma$  is a function of  $\xi$

$$\gamma(\xi) \triangleq \frac{(\xi - 1)}{\epsilon} - q \quad (3.2b)$$

This equation is hyperbolic for  $\beta^2 - b\gamma > 0$  and elliptic for  $\beta^2 - b\gamma < 0$ . Since  $\xi \in [0, 1]$ , a necessary condition for the existence of an elliptic region is  $\beta < 1$  (cf. equations (1.2g) and (1.2h)). The width of such an elliptic region is equal to  $\epsilon(1 - \beta - 2c_t')/2c_t$ ; i.e., it is of the order  $\epsilon$ . Thus, for typical values of  $\epsilon$  ([2]), almost the entire cylinder will lie in the hyperbolic region.

## CHARACTERISTICS IN THE HYPERBOLIC REGION

The dependency of the solution of the almost linear hyperbolic equation (3.2) on the initial conditions ( $\tau = 0$ ) and the boundary conditions (where  $\xi = 0$  or 1), is determined by the geometry of its characteristics. These form a two family net of curves in the  $\xi, \tau$  plane with slopes  $\lambda_1(\xi)$  and  $\lambda_2(\xi)$  satisfying ([9], [10]) the quadratic equation

$$b\gamma\lambda^2 - 2\beta\lambda + 1 = 0 \quad (3.3)$$

That is,

$$\lambda_1 = \frac{\beta - \sqrt{\beta^2 - b\gamma}}{b\lambda} \quad (3.4a)$$

$$\lambda_2 = \frac{\beta + \sqrt{\beta^2 - b\gamma}}{b\gamma} \quad (3.4b)$$

Note that  $\lim_{\gamma \rightarrow 0} \lambda_1 = 1/2\beta$  and  $\lim_{\gamma \rightarrow 0} |\lambda_2| = \infty$ .

A sketch (not exact, but qualitatively correct) of these characteristics appears in Figure 2. Since the domain of dependence of a given point is determined by the region bounded by the characteristics emanating from that point (see Figure 3), the appropriate boundary (initial) conditions for equation (3.2) lie on the axes  $\xi = 0, \tau = 0$ .<sup>†</sup> More precisely, we must specify three functions

$$\phi_1(\tau) = \eta(0, \tau) \quad (3.5a)$$

$$\psi_1(\xi) = \eta(\xi, 0) \quad (3.5b)$$

$$\psi_2(\xi) = \eta_\xi(0, \xi) \quad (3.5c)$$

---

<sup>†</sup>Recall that in this section, we make the assumption  $c_t' \leq 1/2$  so that the end of the cable,  $\xi = 1$ , lies to the right of  $\gamma = 0$ . In the contrary case, we have the situation pictured in Figure 4, and the domain of dependence includes the boundary  $\xi = 1$ ; i.e., the end point. If this were not true, our computations would be inconsistent with the results of section 2.

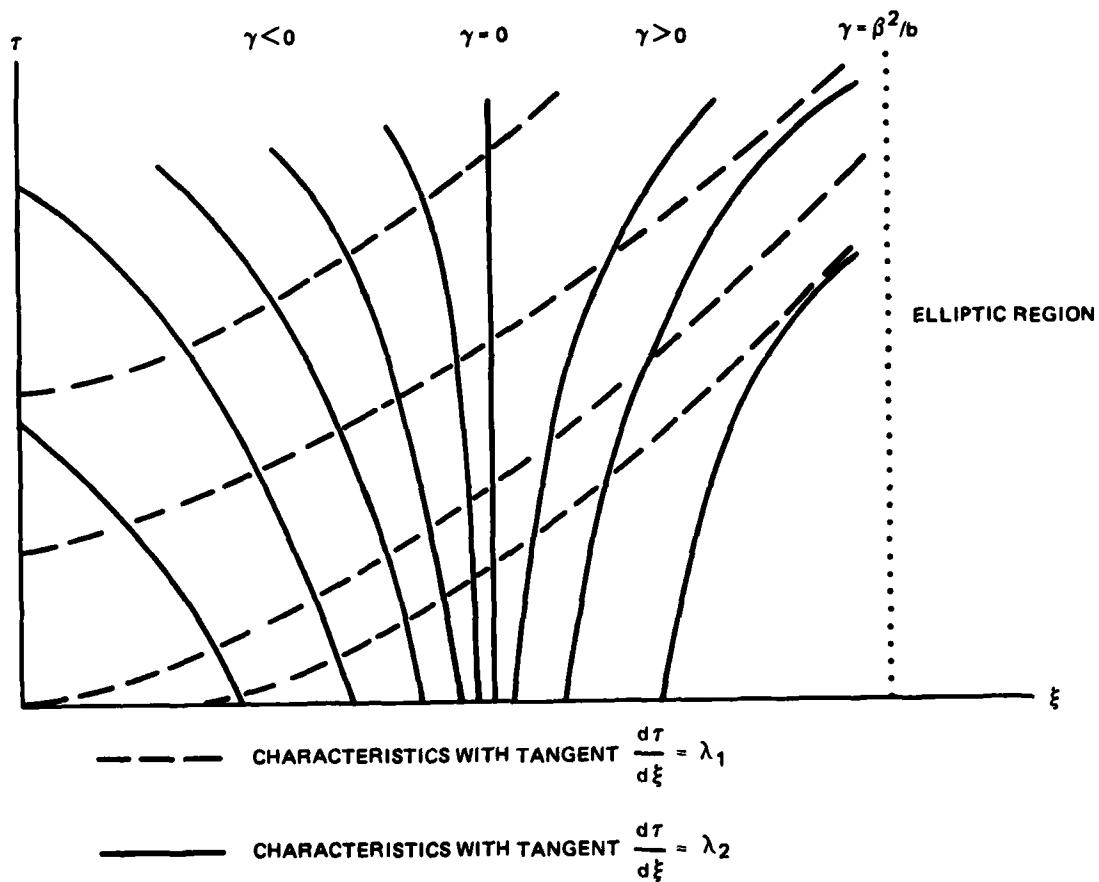


Figure 2. Characteristics of equation (3.2).

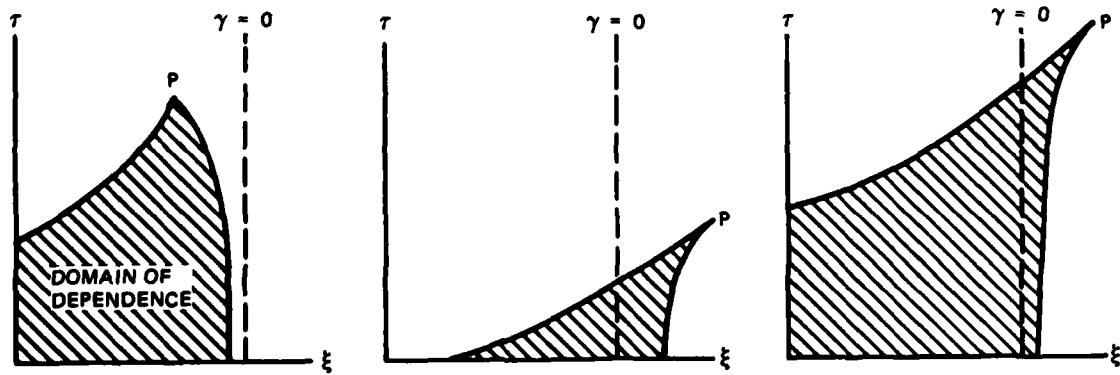


Figure 3. Examples of domains of dependence.

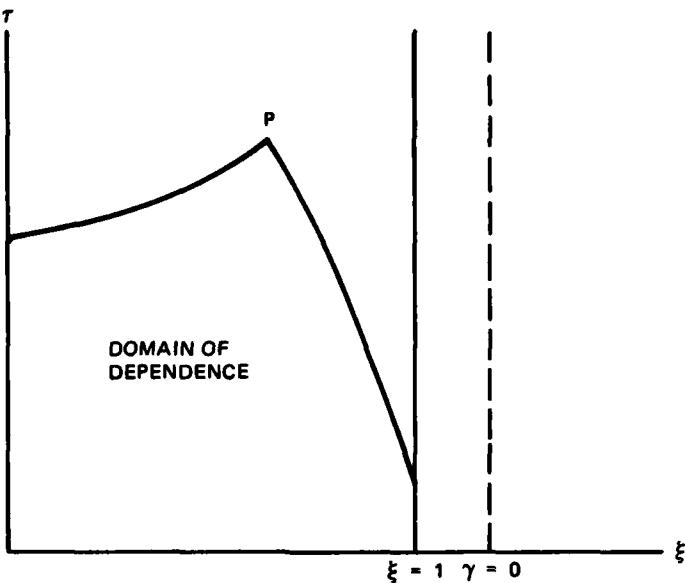


Figure 4. Example showing that the domain of dependence of an interior point P includes the downstream end of the cable if  $c'_t > \frac{1}{2}$  (i.e.,  $\gamma(1) < 0$ ).

This result has several very interesting physical implications. If the solution is stable (i.e., if the transients due to the initial conditions  $\psi_1$  and  $\psi_2$  die out), then the steady state solution depends only on the tow point forcing function  $\phi_1$ . This appears consistent with the results obtained in section II for  $c'_t > \frac{1}{2}$ ; namely, except at the right boundary, the effect of the downstream boundary condition is negligible. Also, the vertical characteristic at  $\gamma = 0$  represents a type of barrier. Past that point, disturbances only propagate downstream. The upstream part of the cylinder ( $\gamma < 0$ ) does not "know" what the downstream end ( $\gamma < 0$ ) is doing. Finally, at  $\gamma = 0$ , we note from equations (3.2b) and (1.1) that

$$MU^2 \frac{\partial^2 y}{\partial x^2} = 2c_t \frac{M}{d_c} U^2(L-x) \frac{\partial^2 y}{\partial x^2}; \quad (3.6)$$

i.e., the transverse force of the fluid momentum exactly balances the transverse component of the tension in the cylinder.<sup>†</sup> At such a point the cylinder should offer vanishing impedance to an external vertical force.

On the other hand, these observations, in turn, generate their own questions. What is so special about the drag coefficient value  $c'_t = \frac{1}{2}$ , and does not the zero impedance at the barrier imply the possible occurrence or persistence of large displacements, i.e., instability?

#### REMARKS ON THE SMALL ANGLE APPROXIMATION

We propose the following answer: If one returns to the original derivation of equation (1.1) [1], it is found that q should actually be given by

$$q = \frac{1}{2c_t} \left( -\frac{1}{\sqrt{1 + |\eta_\xi|^2}} + 2c'_t \right) \quad (3.7)$$

<sup>†</sup>Noted by R. Kennedy, personal communication, Naval Underwater Systems Center.

but that a small angle approximation,  $1/\sqrt{1 + |\eta_\xi|^2} \sim 1$ , was made. With (1.2h),  $\gamma$  vanishes in  $[0,1]$  if and only if  $c'_t \leq \frac{1}{2}$ . On the other hand if expression (3.7) is used, the situation is much more complex since the condition  $\gamma(\xi) = 0$  may depend on the boundary conditions through the term containing  $|\eta_\xi|^2$ . It is still true that  $c'_t < \frac{1}{2}$  represents a sufficient condition for  $\gamma$  to vanish; however, the "barrier" position now becomes a function of time (i.e.,  $\tau$ ). It is also of note that Figure 1 is no longer a valid picture of the characteristics. We cannot even sketch them without further analysis since  $\gamma$  is a function of the solution through  $|\eta_\xi|^2$ . Nevertheless it seems reasonable to assume that in general, there will be no vertical characteristic, and, thus, that a downstream boundary condition must be supplied. This is actually desirable inasmuch as we expect the full fourth order equation to require two downstream boundary conditions, only one of which is lost in going to the reduced PDE. Furthermore, the result will still be consistent with equation (3.2) and its lack of downstream boundary conditions, if we can show that the effects of the nonlinearity (3.7) are restricted to a small region near the boundary; i.e., equation (3.2) and its solution are a good approximation.

To provide a convincing argument (and quantitative bounds) one would have to demonstrate that, given any boundary and initial functions (3.5a) – (3.5c) such that  $|\psi_2|$  is small, then the slope of the solution of (3.2),  $|\eta_\xi|$ , is sufficiently small that it may be neglected in (3.7) except possibly near the boundary. We content ourselves with a heuristic argument concerning the influence of the downstream boundary condition in the nonlinear (eq. (3.7)) case.

Suppose that for all  $\tau$ , the solution  $\eta$  of (3.1) with  $q$  given by (3.7) may be expanded in a Taylor series valid in a region containing  $\xi = 1$  and the curve  $\gamma = 0$  (when it is to the left of  $\xi = 1$ ). Then  $\xi_0 \triangleq 1 - \epsilon(1/c_t)$  is upstream from  $\gamma = 0$ , and  $\eta_\xi(\xi_0) + \eta_\tau(\xi_0) + \alpha\eta_{\tau\tau}(\xi_0) + O(\epsilon) = 0$  by (1.9). However, as seen in section II, the solution upstream of the barrier is of the form  $AF_1 + BF_2$  (equation (2.9)) which must satisfy (2.14) up to order  $\epsilon$  at  $z_1^2 = b(\xi_0 - 1 - \epsilon q)$ . We then find (equations (2.16) – (2.23)) that  $|BF_2|/|AF_1| = O(\epsilon)$  in  $[0, \xi_0]$ , i.e., outside a region of order  $\epsilon$  of the right hand boundary. (The crucial issues in the above handwaving are, of course, the Taylor series and the magnitude of the coefficient of  $\epsilon$  in  $O(\epsilon)$ . If we try to consider the effect of  $\epsilon \rightarrow 0$ , we face the difficult question of the dependency of the series on  $\epsilon$  through equation (3.2)!)!

## CONCLUSIONS

It would appear that, as was the case with  $c'_t > \frac{1}{2}$ , the steady state solution for small flexural rigidity is essentially determined by the single boundary condition  $\eta(0, \tau)$ . This is certainly true for equation (3.1) if it is stable; however, neither the validity of (3.1) as a physical model nor its exact relation to equation (1.2a) has been rigorously established. The mathematical difficulties that one encounters stem from the presence of a singular point in the reduced equation and thus are not alleviated by a perturbation analysis of the fourth order equation.

In general the following open questions remain:

(1) What are the stability properties of the various equations studied in this report? Are the solutions stable with respect to small changes in the coefficients (e.g.,  $c'_t$ )? This also leads to questions concerning the appropriate class of functions for initial conditions and the validity of separation of variables.

(2) What is the significance, if any, of the elliptic region? Does it simply represent a failure of the model due to the approximation  $q = (2c'_t - 1)/2c_t$ ?

(3) From a purely mathematical viewpoint, what is the relationship between equations (1.2) and (3.1) for  $c_t' \leq \frac{1}{2}$ ; can one perform a turning point singular perturbation analysis, and does the limiting solution approach the solution of (3.1)?

Finally, we would like to suggest that a reformulation of the problem with the arclength along the cylinder replacing  $x$  as an independent variable will probably lead to a linear problem and at the same time alleviate many of the above difficulties since it eliminates the small angle approximation. Note that it would also remove the inconsistency of placing the cylinder boundary at  $x = L$  whereas the actual boundary must be at a point  $x < L$  dependent on the shape of the cylinder at the particular point in time. Such an inconsistency could easily be the cause of a spurious elliptical region.

## APPENDIX A

### SEPARATION OF VARIABLES: REDUCED EQUATION

Substituting  $\eta = v(\xi)e^{i\omega\tau}$  into equation (1.2a) we obtain

$$\epsilon^3 hv(4) + b(\xi - 1 - \epsilon q)v(2) + (b + d + 2\beta\epsilon i\omega)v(1) + (i\omega d - \epsilon\omega^2)v = 0 . \quad (\text{A.1})$$

The reduced equation ( $h = 0$ ) may be solved by the substitution ([2])

$$z^2 = b(\xi - 1 - \epsilon q) \quad (\text{A.2})$$

which yields

$$v_{zz} + \frac{v_z}{z} (-1 + \frac{2}{b} (b + d + 2\beta\epsilon i\omega)) + v \left( \frac{4}{b^2} \right) (i\omega d - \epsilon\omega^2) = 0 . \quad (\text{A.3})$$

Let

$$\nu = -\frac{d}{b} - 2\beta\epsilon \frac{i\omega}{b} \quad (\text{A.4})$$

$$b_0^2 = \frac{4\omega}{b} \left( \frac{id}{b} - \epsilon \frac{\omega}{b} \right) . \quad (\text{A.5})$$

Then (A.3) has the two linearly independent solutions ([8], pg. 97)

$$F_1(b_0^2 z^2) \equiv z^\nu J_{-\nu}(b_0 z) \quad (\text{A.6})$$

$$F_2(b_0^2 z^2) \equiv z^\nu J_\nu(b_0 z) \quad (\text{A.7})$$

where

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left( \frac{1}{2} z \right)^{\nu+2m}}{m! \Gamma(\nu + m + 1)} \quad (\text{A.8})$$

is a Bessel function of the first kind. (We have assumed for simplicity that  $\nu$  is not an integer; otherwise we must also introduce the Bessel functions  $Y_\nu$  of the second kind.)

The functions  $F_1$  and  $F_2$  thus have power series

$$F_1(z^2) = \sum_{m=0}^{\infty} \frac{(-1)^m \left( \frac{1}{4} z^2 \right)^m}{m! \Gamma(-\nu + m + 1)} \quad (\text{A.9})$$

$$F_2(z^2) = \sum_{m=0}^{\infty} \frac{(-1)^m \left( \frac{1}{4} z^2 \right)^m \left( \frac{1}{4} z^2 \right)^\nu}{m! \Gamma(\nu + m + 1)} . \quad (\text{A.10})$$

Their first and second derivatives with respect to  $z^2$  are given by

$$F'_1(z^2) = \sum_{m=1}^{\infty} \frac{(-1)^m \frac{1}{4} \left(\frac{1}{4} z^2\right)^{m-1}}{(m-1)! \Gamma(-\nu + m + 1)} \quad (\text{A.11a})$$

$$F''_1(z^2) = \sum_{m=2}^{\infty} \frac{(-1)^m \frac{1}{16} \left(\frac{1}{4} z^2\right)^{m-2}}{(m-2)! \Gamma(-\nu + m + 1)} \quad (\text{A.11b})$$

$$F'_2(z^2) = \sum_{m=0}^{\infty} \frac{(-1)^m \frac{1}{4} (m+\nu) \left(\frac{1}{4} z^2\right)^{m+\nu-1}}{m! \Gamma(\nu + m + 1)} \quad (\text{A.12a})$$

$$F''_2(z^2) = \sum_{m=0}^{\infty} \frac{(-1)^m \frac{1}{16} (m+\nu)(m+\nu-1) \left(\frac{1}{4} z^2\right)^{m+\nu-2}}{m! \Gamma(\nu + m + 1)}. \quad (\text{A.12b})$$

### THE EQUATION FOR $R_0 e^{-g/\mu}$

For simplicity of computation, we make the change of variable

$$x = \xi - 1 - \epsilon q \quad (\text{A.13})$$

and write

$$v(x) = R(x)e^{-g(x)/\mu} \quad (\text{A.14})$$

Substitution of (A.14) into (2.3) requires the derivatives of  $v(x)$  which are given by

$$v' = [R' - g'R/\mu] e^{-g/\mu} \quad (\text{A.15})$$

$$v'' = \left[ R'' - \frac{2g'R'}{\mu} - \frac{g''R}{\mu} + \frac{g'^2R}{\mu^2} \right] e^{-g/\mu} \quad (\text{A.16})$$

$$\begin{aligned} v^{(3)} = & \left[ R^{(3)} - \frac{3g'R''}{\mu} - \frac{3g''R'}{\mu} - \frac{g^{(3)}R}{\mu} \right. \\ & \left. + \frac{3(g')^2R'}{\mu^2} + \frac{3g'g''R}{\mu^2} - \frac{(g')^3R}{\mu^3} \right] e^{-g/\mu} \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} v^{(4)} = & \left[ R^{(4)} - \frac{4g'R^{(3)}}{\mu} - \frac{6g''R''}{\mu} - \frac{4g^{(3)}R'}{\mu} - \frac{g^{(4)}R}{\mu} \right. \\ & \left. + \frac{6(g')^2R''}{\mu^2} + \frac{12g'g''R'}{\mu^2} + \frac{3(g'')^2R}{\mu^2} + \frac{4g'g^{(3)}R}{\mu^2} \right] e^{-g/\mu} \end{aligned}$$

$$- \left[ \frac{4(g')^3 R'}{\mu^3} - \frac{6(g')^2 g'' R}{\mu^3} + \frac{(g')^4 R}{\mu^4} \right] e^{-g/\mu} . \quad (\text{A.18})$$

Substituting these expressions into (2.3), we obtain terms in  $\mu^{-2}$ ,  $\mu^{-1}$ ,  $\mu^0$ ,  $\mu$  and  $\mu^2$ . The coefficient of  $\mu^{-2}$  is then set equal to zero; namely,

$$(g')^4 R_j e^{-g/\mu} + \alpha_1 x (g')^2 R_j e^{-g/\mu} = 0 \quad (\text{A.19})$$

or

$$g' = \pm \sqrt{-\alpha_1 x} \quad (\text{A.20})$$

which has the solution

$$g = \pm \frac{2}{3} \sqrt{\alpha_1} (-x)^{3/2} + \text{constant} . \quad (\text{A.21})$$

The function  $R_0(x)$  is determined by equating the coefficient of the next highest term,  $\mu^{-1} e^{-g/\mu}$ , to zero:

$$-4(g')^3 R'_0 - 6(g')^2 g'' R_0 - 2\alpha_1 x g' R'_0 - \alpha_1 x g'' R_0 - \alpha_2 g' R_0 = 0 . \quad (\text{A.22})$$

Substituting (A.20) into (A.22), we find

$$-2(-x)^{3/2} R'_0 + \left( \frac{5}{2} - \frac{\alpha_2}{\alpha_1} \right) \sqrt{-x} R_0 = 0 . \quad (\text{A.23})$$

Finally, the integral of (A.23) is (since  $x < 0$ )

$$R_0 = r_0(-x) \frac{1}{2} \left( \frac{\alpha_2}{\alpha_1} - \frac{5}{2} \right) . \quad (\text{A.24})$$

The equation for  $L_0$  is the same as that for  $R_0$ ; thus,

$$L_0 = l_0(-x) \frac{1}{2} \left( \frac{\alpha_2}{\alpha_1} - \frac{5}{2} \right) . \quad (\text{A.25})$$

## REFERENCES

1. M. P. Paidoussis, "Dynamics of Flexible Slender Cylinders in Axial Flow," Journal of Fluid Mechanics, Vol 26, 1966, pp 717-751.
2. R. Kennedy and E. Strahan, "A Linear Theory of Transverse Cable Dynamics at Low Frequencies," Naval Underwater Systems Center TR 6463, June 1981.
3. C. R. Ortloff and J. Ives, "On the Dynamic Motion of a Thin Flexible Cylinder in a Viscous Stream," Journal of Fluid Mechanics, Vol 38, part 4, 1969, pp 713-720.
4. R. M. Kennedy, "Low Frequency Transverse Motion Response of Cable Towed Array Systems," NUSC Technical Memorandum No. 93231, Naval Underwater Systems Center, New London, CT, July 1979.
5. R. O'Malley, Jr., Introduction to Singular Perturbations, Academic Press, NY, 1974, Chapters 3 and 8.
6. G. Birkhoff, "On the Asymptotic Character of the Solutions of Certain Linear Differential Equations," Trans. AMS, Vol 9, April 1908, pp 219-230.
7. Turritin, H., "Asymptotic Solutions of Certain Ordinary Differential Equations Associated with Multiple Roots of the Characteristic Equation," Amer. J. Math., Vol 58, 1936, pp 364-378.
8. G. N. Watson, A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge University Press, 1966, p 97.
9. M. J. Shensa, "Characteristic Curves and Normal Form of the Paidoussis Equation," informal memo, Naval Ocean Systems Center, July 1980.
10. G. Forsythe and W. Wasow, Finite-Difference Methods for Partial Differential Equations, John Wiley & Sons, Inc., NY, 1960, pp 38-42.